Int. J. Solids Structures Vol. 23, No. 9, pp. 1285–1300, 1987
Printed in Great Britain.

0020-7683/87 $3.00+ .00

GLOBAL VISCOELASTIC BEHAVIOR OF HETEROGENEOUS THERMOELASTIC MATERIALS

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(*Received* 8 *October* 1986)

Abstract—The global viscoelastic response of heterogeneous linear thermoelastic material with remote boundaries is characterized. Memory effects result from the dissipation of energy due to microscopic temperature gradients. A Fourier transform technique is used to formulate the problem as an integral equation in the image space. Using a perturbation expansion, analytical results are obtained for two- and three-dimensional examples.

1. INTRODUCTION

Thermoelastic damping, i.e. dissipation of mechanical energy due to thermomechanical coupling and heat conduction can be of significant importance in several cases of practical relevance. These include dynamic problems in the form of vibrations[1] or wave propagation[2]. When a macroscopic stress or strain is applied to a heterogeneous thermoelastic medium local stress fluctuations arise. Then, the local temperature gradients generated through thermoelastic coupling result in an irreversible production of entropy. Under these conditions a global viscoelastic behavior is to be expected.

Using a heuristic approach Zener[1] studied the case of polycrystalline materials where the heterogeneity is due to grain-to-grain changes in lattice orientation. General granular media were considered by Buisson *et al.*[3] who formulated the problem in terms of integral equations which were approximately solved by means of a method of spatial discretization. This approach is best suited for granular materials with constant properties within each grain. However, the analysis becomes more complex as the number of grains increases. A one-dimensional problem is solved in Ref. [4] using perturbation expansions.

In this work we are concerned with linear thermoelastic materials of general heterogeneity. A local neighborhood of the body is idealized as being unbounded in all directions and thus Fourier methods can be advantageously utilized. Similar techniques have been successfully used in other studies[5, 6]. The Duhamel and heat equations are transformed into integral equations in the image space. Then, the effective viscoelastic properties of the homogenized continuum follow directly from the inverse integral operators. Explicit results of various orders of approximation can be obtained through a perturbation analysis. This method of approximation has been extensively used in the past. However, the present approach differs from previous work (see, e.g. Refs [7, 8]) in that the fluctuation stresses are described in terms of a stress potential. In addition, the macroscopic or average stress is assumed given. The main benefit which is derived from this strategy is that computation of the first-order term in the perturbation expansion involves purely algebraic manipulations. This is in contrast to displacement formulations based on the use of Green's function which necessitate evaluation of frequently cumbersome integral expressions at any level of approximation. As an example of application of the method, explicit expressions are given for the effective creep functions of two-dimensional thermoelastic bodies.

2. GENERAL FORMULATION

2.1. *Local behavior*

The local behavior of a linear thermoelastic material can be characterized by means of constitutive relations of the Duhamel–Neumann type

$$\sigma_{ij}(\mathbf{x}) = D_{ijkl}(\mathbf{x})\varepsilon_{kl}(\mathbf{x}) - \beta_{ij}(\mathbf{x})\theta(\mathbf{x}) \tag{1}$$

and the heat equation

$$c_{\varepsilon}(\mathbf{x})\dot{\theta}(\mathbf{x}) = (\kappa_{ij}\theta_{,j}(\mathbf{x}))_{,j} - T_0\beta_{ij}(\mathbf{x})\dot{\varepsilon}_{ij}(\mathbf{x}) \tag{2}$$

where σ_{ij} is the Cauchy stress tensor, ε_{ij} the infinitesimal strain tensor, D_{ijkl} the isothermal elastic moduli, β_{ij} the thermal expansion tensor, θ denotes the temperature variation from an initially uniform reference temperature T_0, c_{ε} is the heat capacity at constant deformation and κ_{ij} signifies the thermal conductivity tensor. A subscript comma will be used throughout to denote partial differentiation.

Equivalently, eqn (1) can be inverted to read

$$\varepsilon_{ij} = C_{ijkl}\sigma_{kl} + \alpha_{ij}\theta \tag{3}$$

where C_{ijkl} are the flexibility compliances of the material and one writes

$$\alpha_{ij} = C_{ijkl}\beta_{kl}. \tag{4}$$

Furthermore, elimination of ε_{ij} between eqns (3) and (2) leads to

$$c_{\sigma}\dot{\theta} = (\kappa_{ij}\theta_{,i})_{,j} - T_0\alpha_{kl}\dot{\sigma}_{kl} \tag{5}$$

where

$$c_{\sigma} = c_{\varepsilon} + T_0\beta_{ij}\alpha_{ij} \tag{6}$$

is the heat capacity at constant stress. In addition to the Duhamel and heat equations, eqns (3) and (5), the stress and strain fields must satisfy the equilibrium and compatibility equations.

In formulating the above field equations a general microstructural heterogeneity has been envisioned. This material heterogeneity induces microstructural fluctuation fields which influence the overall response. The detailed nature of these fluctuations is frequently of no practical interest although some information regarding the magnitude of the local fluctuations is sometimes desirable. The analysis that follows aims at providing a link between microstructural properties and the overall response of the material, as well as between the macroscopic fields and the local fluctuations.

We assume throughout that a macroscopic Cauchy stress tensor $\langle\boldsymbol{\sigma}\rangle$ is applied at infinity. The microscopic stress field can be expressed as

$$\sigma_{ij}(\mathbf{x}) = \langle\sigma_{ij}\rangle + \delta\sigma_{ij}(\mathbf{x}) \tag{7}$$

where $\delta\boldsymbol{\sigma}(\mathbf{x})$ represents the spatial stress fluctuations. In the same spirit, the microscopic temperature field can be expressed as the sum of the average temperature $\langle\theta\rangle$ and the temperature fluctuation field $\delta\theta(\mathbf{x})$, i.e.

$$\theta(\mathbf{x}) = \langle\theta\rangle + \delta\theta(\mathbf{x}). \tag{8}$$

The equilibrium equations are automatically satisfied if the fluctuations stresses derive from a stress potential χ, i.e. if

$$\delta\sigma_{ij} = (\nabla\times\chi\times\nabla)_{ij} = \chi_{mn,kl}e_{ikm}e_{jln} \tag{9}$$

where e_{ijk} are the components of the permutation tensor. Substituting eqns (7)–(9) into Duhamel's equation one finds

$$\varepsilon_{ij} = C_{ijkl}:(\langle\sigma_{kl}\rangle+\chi_{mn,pq}e_{pmk}e_{qnl})+\alpha_{ij}(\langle\theta\rangle+\delta\theta). \tag{10}$$

Substituting this expression into the compatibility equations

$$(\nabla\times\varepsilon\times\nabla)_{ij} = \varepsilon_{mn,kl}e_{ikm}e_{jln} = 0 \tag{11}$$

we obtain

$$[C_{klpq}(\chi_{rs,uv}e_{urp}e_{vsp})]_{,mn}e_{mki}e_{nlj}+(\alpha_{kl}\delta\theta)_{,mn}e_{mki}e_{nlj} = -(C_{klrs}\langle\sigma_{rs}\rangle)_{,mn}e_{mki}e_{nlj}-(\alpha_{kl}\langle\theta\rangle)_{,mn}e_{mki}e_{nlj} \tag{12}$$

where we have separated the unknown fluctuation terms from the average values.

Similarly, using eqns (7) and (8) the heat equation becomes

$$c_\sigma\delta\dot{\theta}-(\kappa_{ij}\delta\theta_{,i})_{,j}+T_0\alpha_{ij}\delta\dot{\sigma}_{ij} = -T_0\alpha_{ij}\langle\dot{\sigma}_{ij}\rangle-c_\sigma\langle\dot{\theta}\rangle. \tag{13}$$

This equation can be simplified by invoking the assumed adiabaticity conditions on the remote boundaries. In particular we note that

$$\langle(\kappa_{ij}\theta_{,j})_{,i}\rangle = \lim_{V\to\infty}\frac{1}{V}\int_V(\kappa_{ij}\theta_{,j})_{,i}\,\mathrm{d}V = \lim_{V\to\infty}\frac{1}{V}\int_S q_n\,\mathrm{d}S = 0 \tag{14}$$

since by assumption the normal flux $q_n = \kappa_{ij}\theta_{,i}n_j$ vanishes on remote boundaries. Taking the average value of the heat equation, eqn (5), and using eqn (14) we obtain the relation

$$\langle c_\sigma\dot{\theta}\rangle = -T_0\langle\alpha_{ij}\dot{\sigma}_{ij}\rangle \tag{15}$$

or integrating with respect to time

$$\langle c_\sigma\theta\rangle = -T_0\langle\alpha_{ij}\sigma_{ij}\rangle. \tag{16}$$

Finally, noting that $\langle\delta\theta\rangle = 0$ and $\langle\delta\sigma_{ij}\rangle = 0$ eqn (16) reduces to

$$\langle c_\sigma\rangle\langle\theta\rangle+\langle c_\sigma\delta\theta\rangle+T_0\langle\alpha_{ij}\rangle\langle\sigma_{ij}\rangle+T_0\langle\delta\alpha_{ij}\delta\sigma_{ij}\rangle = 0. \tag{17}$$

For a given macroscopic stress tensor $\langle\sigma\rangle$ eqns (12), (13) and (17) constitute a system of partial differential equations which in principle can be solved for the unknowns χ, $\delta\theta$ and $\langle\theta\rangle$. A method of solution based on the use of Fourier transforms is presented in the next section. A summary of basic facts concerning the Fourier transform which are used in the derivations is given in Appendix A.

2.2. *Localization law*

Taking the Fourier transform of eqn (12) and using eqn (A3), results in the following integral equation:

$$\left(\frac{1}{2\pi}\right)^{3/2}\int e_{irp}e_{jsq}e_{muk}e_{nvl}k_r k_s k'_u k'_v \hat{C}_{pqmn}(\mathbf{k}-\mathbf{k}')\hat{\chi}_{kl}(\mathbf{k}')\,\mathrm{d}^3k'$$

$$-\left(\frac{1}{2\pi}\right)^{3/2}\int e_{imp}e_{jnq}k_m k_n \hat{\alpha}_{pq}(\mathbf{k}-\mathbf{k}')\,\delta\hat{\theta}(\mathbf{k}')\,\mathrm{d}^3k'$$

$$= e_{irp}e_{jsq}k_r k_s \delta\hat{C}_{pqkl}(\mathbf{k})\langle\sigma_{kl}\rangle + e_{irp}e_{jsq}k_r k_s \delta\hat{\alpha}_{pq}(\mathbf{k})\langle\theta\rangle \quad (18)$$

where following standard notation the symbol ^ denotes Fourier transform, i.e.

$$\hat{f}(\mathbf{k}) \equiv \left(\frac{1}{2\pi}\right)^{3/2}\int f(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3x. \quad (19)$$

Similarly, taking the Fourier transform of the heat equation, eqn (13), one finds

$$\left(\frac{1}{2\pi}\right)^{3/2}\int \frac{1}{T_0}\hat{c}_0(\mathbf{k}-\mathbf{k}')\,\delta\dot{\hat{\theta}}(\mathbf{k}')\,\mathrm{d}^3k' + \left(\frac{1}{2\pi}\right)^{3/2}\int \frac{1}{T_0}\hat{\kappa}_{ij}(\mathbf{k}-\mathbf{k}')k_i k'_j \delta\hat{\theta}(\mathbf{k}')\,\mathrm{d}^3k'$$

$$-\left(\frac{1}{2\pi}\right)^{3/2}\int e_{mrp}e_{nsq}k'_r k'_s \hat{\alpha}_{pq}(\mathbf{k}-\mathbf{k}')\dot{\hat{\chi}}_{mn}(\mathbf{k}')\,\mathrm{d}^3k' = -\hat{\alpha}_{kl}(\mathbf{k})\langle\dot{\sigma}_{kl}\rangle - \frac{1}{T_0}\hat{c}_\sigma(\mathbf{k})\langle\dot{\theta}\rangle. \quad (20)$$

Furthermore, taking the Laplace transform of this equation results in the following expression

$$\left(\frac{1}{2\pi}\right)^{3/2}\int \frac{1}{T_0}\hat{c}_\sigma(\mathbf{k}-\mathbf{k}')L\delta\hat{\theta}(\mathbf{k}';s)\,\mathrm{d}^3k' + \left(\frac{1}{2\pi}\right)^{3/2}\int \frac{1}{sT_0}\hat{\kappa}_{ij}(\mathbf{k}-\mathbf{k}')k_i k'_j L\delta\hat{\theta}(\mathbf{k}';s)\,\mathrm{d}^3k'$$

$$-\left(\frac{1}{2\pi}\right)^{3/2}\int e_{mrp}e_{nsq}k'_r k'_s \hat{\alpha}_{pq}(\mathbf{k}-\mathbf{k}')L\hat{\chi}_{mn}(\mathbf{k}';s)\,\mathrm{d}^3k'$$

$$= -\hat{\alpha}_{kl}(\mathbf{k})\langle L\sigma_{kl}(s)\rangle - \frac{1}{T_0}\hat{c}_\sigma(\mathbf{k})\langle L\theta(s)\rangle \quad (21)$$

where the symbol L is used to signify the Laplace transform of a function, i.e.

$$Lf(s) \equiv \int f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t. \quad (22)$$

On the other hand, using eqns (17) and (A3) the mean value of the temperature field can be expressed as

$$\langle\theta\rangle = -\frac{1}{\langle c_\sigma\rangle}\lim_{V\to\infty}\frac{1}{V}\int \hat{c}^*_\sigma(\mathbf{k})\delta\hat{\theta}(\mathbf{k})\,\mathrm{d}^3k - \frac{T_0}{\langle c_\sigma\rangle}\lim_{V\to\infty}\frac{1}{V}\int \hat{\alpha}^*_{ij}(\mathbf{k})\delta\hat{\sigma}_{ij}(\mathbf{k})\,\mathrm{d}^3k$$

$$-\frac{T_0}{\langle c_\sigma\rangle}\langle\alpha_{ij}\rangle\langle\sigma_{ij}\rangle. \quad (23)$$

The limiting process involved in this expression is discussed in Appendix A. Using representation (9), eqn (23) can be rephrased as

$$\langle\theta\rangle = -\frac{1}{\langle c_\sigma\rangle}\lim_{V\to\infty}\frac{1}{V}\int \hat{c}^*_\sigma(\mathbf{k})\delta\hat{\theta}(\mathbf{k})\,\mathrm{d}^3k$$
$$-\frac{T_0}{\langle c_\sigma\rangle}\lim_{V\to\infty}\frac{1}{V}\int \delta\hat{\alpha}^*_{ij}(\mathbf{k})\hat{\chi}_{kl}(\mathbf{k})k_m k_n e_{imk}e_{jnl}\,\mathrm{d}^3k - \frac{T_0}{\langle c_\sigma\rangle}\langle\alpha_{ij}\rangle\langle\sigma_{ij}\rangle. \quad (24)$$

Eliminating $\langle\theta\rangle$ from eqns (18) and (21) with the aid of eqn (24) we arrive at a system of two coupled integral equations for the unknowns χ_{kl} and $\delta\theta$

$$\int K_{ijkl}(\mathbf{k},\mathbf{k}';s)L\hat{\chi}_{kl}(\mathbf{k}';s)\,\mathrm{d}^3k' + \int K_{ij}(\mathbf{k},\mathbf{k}';s)L\delta\hat{\theta}(\mathbf{k}';s)\,\mathrm{d}^3k' = N_{ijkl}(\mathbf{k};s)L\langle\sigma_{kl}\rangle(s)$$
$$\int K^*_{ij}(\mathbf{k}',\mathbf{k};s)L\hat{\chi}_{ij}(\mathbf{k}';s)\,\mathrm{d}^3k' + \int K(\mathbf{k},\mathbf{k}';s)L\delta\hat{\theta}(\mathbf{k}';s)\,\mathrm{d}^3k' = N_{ij}(\mathbf{k};s)L\langle\sigma_{ij}\rangle(s) \quad (25)$$

where the kernels and the forcing terms are defined to be

$$K_{ijkl}(\mathbf{k},\mathbf{k}';s) = \left(\frac{1}{2\pi}\right)^{3/2} e_{irp}e_{jsq}e_{muk}e_{nvl}k_r k_s k'_u k'_v \hat{C}_{pqmn}(\mathbf{k}-\mathbf{k}')$$
$$-\frac{T_0}{\langle c_\sigma\rangle}\delta\hat{\alpha}_{pq}(\mathbf{k})\lim_{V\to\infty}\frac{1}{V}e_{irp}e_{jsq}e_{muk}e_{nvl}k_r k_s k'_u k'_v \delta\hat{\alpha}^*_{mn}(\mathbf{k}')$$

$$K_{ij}(\mathbf{k},\mathbf{k}';s) = -\left(\frac{1}{2\pi}\right)^{3/2} e_{imp}e_{jnq}k_m k_n \hat{\alpha}_{pq}(\mathbf{k}-\mathbf{k}) + \frac{1}{\langle c_\sigma\rangle}\delta\hat{\alpha}_{pq}(\mathbf{k})\lim_{V\to\infty}\frac{1}{V}e_{imp}e_{jnq}k_m k_n \hat{c}^*_\sigma(\mathbf{k}')$$
$$(26)$$

$$K(\mathbf{k},\mathbf{k}';s) = \left(\frac{1}{2\pi}\right)^{3/2}\frac{1}{T_0}\hat{c}_\sigma(\mathbf{k}-\mathbf{k}') + \left(\frac{1}{2\pi}\right)^{3/2}\frac{1}{sT_0}\hat{\kappa}_{ij}(\mathbf{k}-\mathbf{k}')k_i k'_j - \frac{1}{T_0\langle c_\sigma\rangle}\hat{c}_\sigma(\mathbf{k})\lim_{V\to\infty}\frac{1}{V}\hat{c}^*_\sigma(\mathbf{k}')$$

$$N_{ijkl}(\mathbf{k}) = e_{irp}e_{jsq}k_r k_s \delta\hat{C}_{pqkl}(\mathbf{k}) - \frac{T_0}{\langle c_\sigma\rangle}e_{irp}e_{jsq}k_r k_s \delta\hat{\alpha}_{pq}(\mathbf{k})\langle\alpha_{kl}\rangle$$
$$N_{ij}(\mathbf{k}) = -\hat{\alpha}_{ij}(\mathbf{k}) + \frac{\hat{c}_\sigma(\mathbf{k})}{\langle c_\sigma\rangle}\langle\alpha_{ij}\rangle.$$

As can be seen, the integral operator defined by eqns (25) and (26) is hermitian. Under mild restrictions on the fluctuations of the material properties a simple check reveals that the integral operator is of the Hilbert–Schmidt type. The prescribed average stress drives the microscopic fluctuation fields. These vanish identically in the absence of material heterogeneities.

Formally inverting eqn (25) one obtains

$$L\hat{\chi}_{ij}(\mathbf{k};s) = \left\{\int H_{ijkl}(\mathbf{k},\mathbf{k}';s)N_{klmn}(\mathbf{k}')\,\mathrm{d}^3k' + \int H_{ij}(\mathbf{k},\mathbf{k}';s)N_{mn}(\mathbf{k}')\,\mathrm{d}^3k'\right\}L\langle\sigma_{mn}\rangle(s)$$
$$\equiv B_{ijmn}(\mathbf{k};s)L\langle\sigma_{mn}\rangle(s) \quad (27)$$

$$L\delta\hat{\theta}(\mathbf{k};s) = \left\{\int H^*_{kl}(\mathbf{k}',\mathbf{k};s)N_{klmn}(\mathbf{k}')\,\mathrm{d}^3k' + \int H(\mathbf{k},\mathbf{k}';s)N_{mn}(\mathbf{k}')\,\mathrm{d}^3k'\right\}L\langle\sigma_{mn}\rangle(s) \equiv B_{mn}(\mathbf{k};s)L\langle\sigma_{mn}\rangle(s)$$

where B_{ijkl} and B_{ij} are concentration tensors. Equation (27) represents a localization law giving the Fourier–Laplace transform of the fluctuation fields as a function of the applied macroscopic stress.

Remark: It is also possible to utilize a decomposition $\theta = \theta_0 + \delta\tilde{\theta}$ where

$$\theta_0 = -T_0\frac{\langle\alpha_{kl}\rangle\langle\sigma_{kl}\rangle}{\langle c_\sigma\rangle} \tag{28}$$

is the uniform temperature variation generated by the average stresses in a homogeneous reference material having the average thermomechanical properties. It should be noted that $\langle\delta\theta\rangle$ is not necessarily zero. In terms of this alternative decomposition equations similar to eqns (25) are obtained in which $\delta\tilde{\theta}$ replaces $\delta\theta$.

2.3. *Effective behavior*

Taking averages in Duhamel's equation, eqn (3), and using decomposition (7) and (8) for stress and temperature results in the following expression:

$$\langle\varepsilon_{ij}\rangle = \langle C_{ijkl}\rangle\langle\sigma_{kl}\rangle + \langle\alpha_{ij}\rangle\langle\theta\rangle + \langle C_{ijkl}\delta\sigma_{kl}\rangle + \langle\alpha_{ij}\delta\theta\rangle. \tag{29}$$

Eliminating $\langle\theta\rangle$ with the aid of eqn (17) we find

$$\langle\varepsilon_{ij}\rangle = \langle C_{ijkl}\rangle\langle\sigma_{kl}\rangle - \frac{\langle\alpha_{ij}\rangle}{\langle c_\sigma\rangle}\langle c_\sigma\delta\theta\rangle - \frac{T_0}{\langle c_\sigma\rangle}\langle\alpha_{ij}\rangle\langle\alpha_{kl}\sigma_{kl}\rangle - \frac{T_0}{\langle c_\sigma\rangle}\langle\alpha_{ij}\rangle\langle\delta\alpha_{kl}\delta\sigma_{kl}\rangle + \langle C_{ijkl}\delta\sigma_{kl}\rangle + \langle\alpha_{ij}\delta\theta\rangle \tag{30}$$

or, using the identities $\langle\delta\theta\rangle = 0$, $\langle\delta\sigma_{kl}\rangle = 0$

$$\langle\varepsilon_{ij}\rangle = \left(\langle C_{ijkl}\rangle - \frac{T_0}{\langle c_\sigma\rangle}\langle\alpha_{ij}\rangle\langle\alpha_{kl}\rangle\right)\langle\sigma_{kl}\rangle - \frac{\langle\alpha_{ij}\rangle}{\langle c_\sigma\rangle}\langle c_\sigma\delta\theta\rangle - \frac{T_0}{\langle c_\sigma\rangle}\langle\alpha_{ij}\rangle\langle\delta\alpha_{kl}\delta\sigma_{kl}\rangle + \langle\delta C_{ijkl}\delta\sigma_{kl}\rangle + \langle\delta\alpha_{ij}\delta\theta\rangle. \tag{31}$$

Bringing in eqns (A3) this becomes

$$\langle\varepsilon_{ij}\rangle = \left(\langle C_{ijkl}\rangle - \frac{T_0}{\langle c_\sigma\rangle}\langle\alpha_{ij}\rangle\langle\alpha_{kl}\rangle\right)\langle\sigma_{kl}\rangle - \lim_{V\to\infty}\frac{1}{V}\int N^*_{mnij}(\mathbf{k})\hat{\chi}_{mn}(\mathbf{k})\,\mathrm{d}^3k - \lim_{V\to\infty}\frac{1}{V}\int N^*_{ij}(\mathbf{k})\delta\hat{\theta}(\mathbf{k})\,\mathrm{d}^3k \tag{32}$$

where N_{ijkl} and N_{ij} are given by eqns (26e) and (26f). Finally, we take the Laplace transform of this expression and make use of the localization law, eqns (27), to obtain

$$L\langle\varepsilon_{ij}\rangle(s) = \left[\langle C_{ijkl}\rangle - T_0\frac{\langle\alpha_{ij}\rangle\langle\alpha_{kl}\rangle}{\langle c_\sigma\rangle}\right]L\langle\sigma_{kl}\rangle(s) + F_{ijkl}(s)L\langle\sigma_{kl}\rangle(s) \tag{33}$$

where

$$F_{ijkl}(s) = -\lim_{V\to\infty}\frac{1}{V}\int N^*_{mnij}(\mathbf{k})B_{mnkl}(\mathbf{k})\ \mathrm{d}^3k - \lim_{V\to\infty}\frac{1}{V}\int N^*_{ij}(\mathbf{k})B_{kl}(\mathbf{k})\ \mathrm{d}^3k. \tag{34}$$

It is interesting to note that eqn (33) defines a hereditary law of viscoelastic type

$$\langle\varepsilon_{ij}\rangle = J_{ijkl} * \mathrm{d}\langle\sigma_{kl}\rangle \tag{35}$$

where J_{ijkl} are the creep functions, * is the time convolution operation and d signifies differentiation with respect to time in a distributional sense. The form of the creep functions is obtained by taking the Laplace transform of eqn (35)

$$L\langle\varepsilon_{ij}\rangle(s) = sLJ_{ijkl}(s)L\langle\sigma_{kl}\rangle(s) \tag{36}$$

and comparing the result to eqn (34), which yields

$$sLJ_{ijkl}(s) = \left[\langle C_{ijkl}\rangle - T_0\frac{\langle\alpha_{ij}\rangle\langle\alpha_{kl}\rangle}{\langle c_\sigma\rangle}\right] + F_{ijkl}(s). \tag{37}$$

It should be noted that the variable s enters eqns (26) only through the term $\kappa_{ij}(\mathbf{k}-\mathbf{k}')/s$. In the absence of heat conduction this term vanishes identically and rheological effects disappear.

In conclusion, the effective behavior of a linear thermoelastic body with adiabatic remote boundaries and heterogeneous microstructure corresponds to that of a linear viscoelastic material. The effective creep functions have been given in a form whose evaluation requires inversion of an integral operator. To make further progress the inverse operator can be approximated by means of a perturbation expansion in the expectation that the leading terms will provide an adequate description of cases involving mild heterogeneities. As noted in the introduction, an advantage of the method presented here is that the first term in the expansion is given by algebraic expressions of a particularly simple form. In subsequent sections, explicit results are obtained from the general formulation for certain classes of two-dimensional problems.

3. TWO-DIMENSIONAL ISOTROPIC PROBLEM

3.1. *Local behavior*

The general equations derived in Section 2 simplify considerably for a two-dimensional, locally isotropic body. Under these conditions, the spatial distribution of elastic properties can be defined in terms of the local Young's modulus E and Poisson's ratio ν, and the elastic compliances are of the form

$$C_{\alpha\beta\gamma\delta}(\mathbf{x}) = \tfrac{1}{2}(f(\mathbf{x})+g(\mathbf{x}))\,(\delta_{\alpha\gamma}\,\delta_{\beta\delta}+\delta_{\alpha\delta}\,\delta_{\beta\gamma}) - g(\mathbf{x})\,\delta_{\alpha\beta}\,\delta_{\gamma\delta} \tag{38}$$

where Greek indices range from 1 to 2 and the scalar functions $f(\mathbf{x})$ and $g(\mathbf{x})$ are defined as

$$f = \frac{1}{E}, \qquad g = \frac{\nu}{E} \tag{39}$$

for plane stress, and

$$f = (1-\nu^2)/E, \qquad g = \nu(1+\nu)/E \tag{40}$$

for plane strain. Furthermore, the thermal expansion and conductivity tensors take the form

$$\alpha_{ij} = \alpha\delta_{ij}, \qquad \kappa_{ij} = \kappa\delta_{ij} \tag{41}$$

and the Duhamel law and heat equation read

$$\varepsilon_{\alpha\beta} = C_{\alpha\beta\gamma\delta}\sigma_{\gamma\delta} + \delta_{\alpha\beta}h\theta \tag{42}$$

$$c\dot{\theta} = (\kappa\theta_{,\alpha})_{,\alpha} - T_0 h\dot{\sigma}_{\alpha\alpha} \tag{43}$$

where one writes

$$h = \alpha, \qquad c = c_\sigma \tag{44}$$

for plane stress, and

$$h = (1+\nu)\alpha, \qquad c = c_\sigma - T_0\alpha^2 E \tag{45}$$

for plane strain.

In what follows, the above equations are rendered dimensionless by introducing normalized space–time variables

$$\begin{aligned} t &= \frac{L^2\langle c\rangle}{\langle\kappa\rangle}\bar{t} \\ x_i &= L\bar{x}_i \end{aligned} \tag{46}$$

where L is some characteristic length of the heterogeneities, and setting

$$\begin{aligned} \bar{\sigma}_{\alpha\beta} &= \langle f\rangle\sigma_{\alpha\beta} \\ \bar{\theta} &= \langle h\rangle\theta \end{aligned} \tag{47}$$

for stresses and temperature and

$$\begin{aligned} \bar{C}_{\alpha\beta\gamma\delta} &= C_{\alpha\beta\gamma\delta}/\langle f\rangle \\ \bar{h} &= h/\langle h\rangle \end{aligned} \tag{48}$$

for the elastic compliances and thermal expansion coefficient. In terms of dimensionless variables, the Duhamel law, eqn (42), and the heat equation, eqn (43), can be expressed as

$$\begin{aligned} \varepsilon_{\alpha\beta} &= \bar{C}_{\alpha\beta\gamma\delta}\bar{\sigma}_{\gamma\delta} + \delta_{\alpha\beta}\bar{h}\bar{\theta} \\ \bar{c}\dot{\bar{\theta}} &= (\bar{\kappa}\bar{\theta}_{,\alpha})_{,\alpha} - \varepsilon\bar{h}\dot{\bar{\sigma}}_{\alpha\alpha} \end{aligned} \tag{49}$$

with

$$\begin{aligned} \bar{c} &= c/\langle c\rangle, \qquad \bar{\kappa} = \kappa/\langle\kappa\rangle \\ \varepsilon &= \frac{T_0\langle h\rangle^2}{\langle c\rangle\langle f\rangle}. \end{aligned} \tag{50}$$

The coupling parameter ε is usually a small quantity ($\varepsilon \ll 1$). Equations (49) are formally obtained from eqns (42) and (43) by replacing all variables by their dimensionless counterpart and T_0 by ε.

In what follows, we focus on the plane stress case. Passage to plane strain can be accomplished by making the substitutions

$$E \rightarrow \frac{E}{1-\nu^2}, \qquad \nu \rightarrow \frac{\nu}{1-\nu}, \qquad \alpha \rightarrow (1+\nu)\alpha.$$

3.2. *Localization law*

In the plane case, eqns (25) and (26) simplify considerably as a result of the fact that all quantities are x_3-independent and that the only non-zero component of the stress potential is χ_{33}, henceforth denoted by χ. For simplicity of notation, the superimposed bar on all dimensionless quantities is henceforth omitted. Furthermore, we use eqns (48) and (50) to conclude that

$$\langle h \rangle = 1, \qquad \langle c \rangle = 1, \qquad \langle \kappa \rangle = 1. \tag{51}$$

With this simplification, eqns (25) and (26) reduce to

$$\begin{gathered}\int K^c(\mathbf{k},\mathbf{k}',s) L\hat{\chi}(\mathbf{k}',s)\,\mathrm{d}^2k' + \int K^\alpha(\mathbf{k},\mathbf{k}',s) L\delta\hat{\theta}(\mathbf{k}',s)\,\mathrm{d}^2k' = \delta\tilde{N}_{\alpha\beta}(\mathbf{k}) L\langle\sigma_{\alpha\beta}\rangle(s) \\ \int K^{\alpha *}(\mathbf{k}',\mathbf{k},s) L\hat{\chi}(\mathbf{k}',s)\,\mathrm{d}^2k' + \int K(\mathbf{k},\mathbf{k}',s) L\delta\hat{\theta}(\mathbf{k}',s)\,\mathrm{d}^2k' = N(\mathbf{k}) L\langle\sigma_{\alpha\alpha}\rangle(s)\end{gathered} \tag{52}$$

where the kernels and the forcing terms are given by

$$\begin{gathered}K^c(\mathbf{k},\mathbf{k}',s) = \frac{1}{2\pi} A(\mathbf{k},\mathbf{k}')\hat{f}(\mathbf{k}-\mathbf{k}') - \frac{1}{2\pi} B(\mathbf{k},\mathbf{k}')\hat{g}(\mathbf{k}-\mathbf{k}') - \varepsilon k^2\,\delta\hat{h}(\mathbf{k}) \lim_{S\rightarrow\infty}\frac{1}{S} k'^2\,\delta\hat{h}^*(\mathbf{k}') \\ A(\mathbf{k},\mathbf{k}') = (k_1k_1'+k_2k_2')^2, \qquad B(\mathbf{k},\mathbf{k}') = (k_1k_2'-k_2k_1')^2, \\ K^\alpha(\mathbf{k},\mathbf{k}',s) = -\frac{1}{2\pi}k^2\hat{h}(\mathbf{k}-\mathbf{k}') + k^2\,\delta\hat{h}(\mathbf{k}) \lim_{S\rightarrow\infty}\frac{1}{S}\hat{c}^*(\mathbf{k}') \\ K(\mathbf{k},\mathbf{k}',s) = \frac{1}{2\pi}\frac{1}{\varepsilon}\hat{c}(\mathbf{k}-\mathbf{k}') - \frac{\hat{c}(k)}{\varepsilon}\lim_{S\rightarrow\infty}\frac{1}{S}\hat{c}^*(\mathbf{k}') + \frac{1}{2\pi}\frac{1}{\varepsilon s}\hat{\kappa}(\mathbf{k}-\mathbf{k}')k_\alpha' k_\alpha \\ \delta\tilde{N}_{\alpha\beta}(\mathbf{k}) = (\delta\hat{f}(\mathbf{k}) - \varepsilon\delta\hat{h}(\mathbf{k}))k^2\,\delta_{\alpha\beta} - (\delta\hat{f}(\mathbf{k}) + \delta\hat{g}(\mathbf{k}))k_\alpha k_\beta, \quad N(\mathbf{k}) = (\hat{c}(\mathbf{k}) - \hat{h}(\mathbf{k}))\,\delta_{\alpha\beta}\end{gathered} \tag{53}$$

and we write $k^2 = k_1^2 + k_2^2$. In deriving the above expressions, use has been made of the identity

$$\lim_{V\rightarrow\infty}\frac{1}{V}\int \hat{c}_\sigma^*(k_1',k_2',k_3')\,\delta\hat{\theta}(k_1',k_2',k_3')\,\mathrm{d}^3k' = \langle c_\sigma\delta\theta\rangle = \lim_{S\rightarrow\infty}\frac{1}{S}\int \hat{c}_\sigma^*(k_1',k_2')\,\delta\hat{\theta}(k_1',k_2')\,\mathrm{d}^2k'$$

where in the first integral we consider three-dimensional Fourier transforms, whereas in the last term the Fourier transforms are two-dimensional. Thus, the volume average $\langle\ \rangle$ reduces to the plane mean value as a result of the fact that the quantities being averaged are independent of x_3.

To explicate the solution further, we make the additional assumption that fluctuations in the material parameters are small, i.e.

$$\delta f = O(\varepsilon_1), \qquad \delta g = O(\varepsilon_1), \qquad \delta h = O(\varepsilon_1), \qquad \delta c = O(\varepsilon_1) \tag{54}$$

where $\varepsilon_1 \ll 1$ is a small parameter.

By virtue of assumption (54) system (52) involves two small parameters, ε_1 and the thermomechanical coupling constant ε. Next we seek a solution $(\chi, \delta\theta)$ of system (52) of the form

$$\begin{aligned} \chi &= \chi_{10}+\chi_{01}+\chi_{11}+\chi_{20}+\chi_{02}+\cdots \\ \delta\theta &= \delta\theta_{11}+\delta\theta_{20}+\delta\theta_{02}+\cdots \end{aligned} \tag{55}$$

where the terms of the expansion are characterized by the conditions

$$\chi_{ij} = O(\varepsilon_1^i \varepsilon^j), \qquad \delta\theta_{ij} = O(\varepsilon_1^i \varepsilon^j).$$

Substituting expansion (55) into system (52) and gathering terms of same order we obtain

$$\begin{aligned} L\delta\hat{\theta}(\mathbf{k},s) &= sL\hat{\Omega}_{\alpha\beta}(\mathbf{k},s)L\langle\sigma_{\alpha\beta}\rangle(s)+O(\varepsilon\varepsilon_1^2)+O(\varepsilon_1^3) \\ k^2L\hat{\chi}(\mathbf{k},s) &= sL\hat{\Gamma}_{\alpha\beta}(\mathbf{k},s)L\langle\sigma_{\alpha\beta}\rangle(s)+O(\varepsilon_1^3)+O(\varepsilon\varepsilon_1^2) \end{aligned} \tag{56}$$

where the localization tensors $\Omega_{\alpha\beta}$ and $\Gamma_{\alpha\beta}$ are given by

$$L\hat{\Omega}_{\alpha\beta}(\mathbf{k},s) = \frac{\varepsilon}{s}\left(1+\frac{k^2}{s}\right)^{-1}\left[\frac{\delta N^0_{\alpha\beta}}{k^2}(\mathbf{k})+\delta_{\alpha\beta}\,\delta\hat{c}(\mathbf{k})-\delta_{\alpha\beta}\,\delta\hat{h}(\mathbf{k})\right] \tag{57}$$

$$\begin{aligned} L\hat{\Gamma}_{\alpha\beta}(\mathbf{k},s) = \frac{1}{s}\Bigg\{\frac{\delta\tilde{N}_{\alpha\beta}(\mathbf{k})}{k^2} - \frac{1}{2\pi k^2}\int\frac{1}{k'^4}\Big[&\delta\hat{f}(\mathbf{k}-\mathbf{k}')A(\mathbf{k},\mathbf{k}') \\ &-\delta\hat{g}(\mathbf{k}-\mathbf{k}')B(\mathbf{k},\mathbf{k}')\Big]\delta N^0_{\alpha\beta}(\mathbf{k}')\,\mathrm{d}^2k'\Bigg\}+L\hat{\Omega}_{\alpha\beta}(\mathbf{k},s) \end{aligned}$$

with $\delta N^0_{\alpha\beta} = \delta\tilde{N}_{\alpha\beta}$ at $\varepsilon = 0$; i.e.

$$\delta N^0_{\alpha\beta}(\mathbf{k}) = \delta\hat{f}(\mathbf{k})k^2\,\delta_{\alpha\beta}-(\delta\hat{f}(\mathbf{k})+\delta\hat{g}(\mathbf{k}))k_\alpha k_\beta. \tag{58}$$

3.3. *Effective behavior*

The effective behavior is obtained by particularizing the three-dimensional law, eqn (33), to plane conditions

$$\langle\varepsilon_{\alpha\beta}\rangle(t) = J_{\alpha\beta\gamma\delta} * \mathrm{d}\langle\sigma_{\gamma\delta}\rangle(t) \tag{59}$$

or, in terms of Laplace transforms

$$L\langle\varepsilon_{\alpha\beta}\rangle(s) = sLJ_{\alpha\beta\gamma\delta}(s)L\langle\sigma_{\gamma\delta}\rangle(s). \tag{60}$$

Here, the creep functions are given by

$$\begin{aligned} sLJ_{\alpha\beta\gamma\delta}(s) &= \langle c_{\alpha\beta\gamma\delta}\rangle-\varepsilon\,\delta_{\alpha\beta}\,\delta_{\gamma\delta} \\ &\quad-\lim_{S\to\infty}\frac{1}{S}\Bigg\{\delta_{\alpha\beta}\left(\int\delta\hat{c}^*(\mathbf{k})-\delta\hat{h}^*(\mathbf{k})sL\hat{\Omega}_{\gamma\delta}(\mathbf{k},s)\,\mathrm{d}^2k-\varepsilon\int\delta\hat{h}^*(\mathbf{k})\frac{\delta N^0_{\gamma\delta}(\mathbf{k})}{k^2}\,\mathrm{d}^2k\right) \\ &\qquad+\int\delta N^{0*}_{\alpha\beta}(\mathbf{k})s\frac{L\hat{\Gamma}_{\gamma\delta}(\mathbf{k},s)}{k^2}\,\mathrm{d}^2k\Bigg\}+O(\varepsilon\varepsilon_1^3)+O(\varepsilon_1^4) \end{aligned} \tag{61}$$

in image space, or

$$
\begin{aligned}
J_{\alpha\beta\gamma\delta}(t) = {} & \langle c_{\alpha\beta\gamma\delta}\rangle - \varepsilon\delta_{\alpha\beta}\,\delta_{\gamma\delta} + \varepsilon\delta_{\alpha\beta}\lim_{S\to\infty}\frac{1}{S}\int \delta\hat{h}^{*}(\mathbf{k})\frac{\delta N^{0}_{\gamma\delta}(\mathbf{k})}{k^{2}}\,\mathrm{d}^{2}k \\
& - \lim_{S\to\infty}\frac{1}{S}\int\frac{\delta N^{0*}_{\alpha\beta}(\mathbf{k})}{k^{2}}\left\{\frac{\delta N_{\gamma\delta}(\mathbf{k})}{k^{2}} - \frac{1}{2\pi k^{2}}\int\frac{1}{k'^{4}}[\delta\hat{f}(\mathbf{k}-\mathbf{k}')A(\mathbf{k},\mathbf{k}')\right. \\
& \left. - \delta\hat{g}(\mathbf{k}-\mathbf{k}')B(\mathbf{k},\mathbf{k}')]\,\delta N^{0}_{\gamma\delta}(\mathbf{k}')\,\mathrm{d}^{2}k'\right\}\mathrm{d}^{2}k - \varepsilon\lim_{S\to\infty}\frac{1}{S}\int\left[\delta_{\alpha\beta}\delta\hat{c}^{*}(\mathbf{k}) - \delta_{\alpha\beta}\delta\hat{h}^{*}(\mathbf{k})\right. \\
& \left. + \frac{\delta N^{0*}_{\alpha\beta}(\mathbf{k})}{k^{2}}\right]\left[\delta_{\gamma\delta}\delta\hat{c}(\mathbf{k}) - \delta_{\gamma\delta}\delta\hat{h}(\mathbf{k}) + \frac{\delta N^{0}_{\gamma\delta}(\mathbf{k})}{k^{2}}\right]\mathrm{e}^{-k^{2}t}\,\mathrm{d}^{2}k + O(\varepsilon_{1}^{4}) + O(\varepsilon\varepsilon_{1}^{3})
\end{aligned}
\tag{62}
$$

in terms of time.

This expression can be recast as

$$
J_{\alpha\beta\gamma\delta}(t) = J_{\alpha\beta\gamma\delta}(\infty) - \varepsilon\lim_{S\to\infty}\frac{1}{S}\int A_{\alpha\beta\gamma\delta}(\mathbf{k})\,\mathrm{e}^{-k^{2}t}\,\mathrm{d}^{2}k \tag{63}
$$

where $J_{\alpha\beta\gamma\delta}(\infty)$ is the long range creep tensor, and

$$
A_{\alpha\beta\gamma\delta}(\mathbf{k}) = \left[\delta_{\alpha\beta}\delta\hat{c}^{*}(\mathbf{k}) - \delta_{\alpha\beta}\delta\hat{h}^{*}(\mathbf{k}) + \frac{\delta N^{0*}_{\alpha\beta}(\mathbf{k})}{k^{2}}\right]\left[\delta_{\gamma\delta}\delta\hat{c}(\mathbf{k}) - \delta_{\gamma\delta}\delta\hat{h}(\mathbf{k}) + \frac{\delta N^{0}_{\gamma\delta}(\mathbf{k})}{k^{2}}\right]. \tag{64}
$$

In the absence of thermomechanical coupling, $\varepsilon = 0$ and viscoelastic effects are not present in the material. For $\varepsilon > 0$, the creep functions $J_{\alpha\beta\gamma\delta}$ are seen to have a continuous relaxation spectrum. Furthermore, the scalars $A_{\alpha\beta\alpha\beta}$ (no sum on α or β) are real and positive. Hence, $J_{\alpha\beta\alpha\beta}$ is a Bernstein function[9]. The derivatives of $J_{\alpha\beta\alpha\beta}$ are alternating, i.e.

$$
J_{\alpha\beta\alpha\beta}(t) \geqslant 0
$$

and for $n \geqslant 1$

$$
(-1)^{n+1}J^{(n)}_{\alpha\beta\alpha\beta}(t) \geqslant 0 \quad \text{(no sum on } \alpha \text{ or } \beta\text{)}. \tag{65}
$$

The instantaneous compliances $J_{\alpha\beta\alpha\beta}(0)$ of the material can be calculated by setting $t = 0$ in eqn (62) or eqn (63). Alternatively, one can proceed as follows. Since sufficiently fast loading is adiabatic, the stress–strain relation reads

$$
\varepsilon_{\alpha\beta} = C^{(\mathrm{ad})}_{\alpha\beta\gamma\delta}\sigma_{\gamma\delta}
$$

where the adiabatic compliances are given by

$$
C^{(\mathrm{ad})}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} - \frac{\varepsilon h^{2}}{c}\delta_{\alpha\beta}\,\delta_{\gamma\delta} = \tfrac{1}{2}(\tilde{f}+\tilde{g})\,(\delta_{\alpha\gamma}\,\delta_{\beta\delta} + \delta_{\alpha\delta}\,\delta_{\beta\gamma}) - \tilde{g}\delta_{\alpha\beta}\,\delta_{\gamma\delta} \tag{66}
$$

with

$$\tilde{f} = f - \varepsilon\frac{h^2}{c}, \qquad \tilde{g} = g + \frac{\varepsilon h^2}{c}. \tag{67}$$

For an elastic material with no thermoelastic coupling, the effective compliances are given by eqn (62) with $\varepsilon = 0$. Considering the material defined by expression (65) we get from eqn (62) by replacing $C_{\alpha\beta\gamma\delta}$ by $C^{(\mathrm{ad})}_{\alpha\beta\gamma\delta}$, f and g by $\tilde{f}$ and $\tilde{g}$ and putting $\varepsilon = 0$

$$J_{\alpha\beta\gamma\delta}(0) = \langle C^{\mathrm{ad}}_{\alpha\beta\gamma\delta}\rangle - \lim_{S\to\infty}\frac{1}{S}\int\frac{\delta\tilde{N}^*_{\alpha\beta}(\mathbf{k})}{\langle\tilde{f}\rangle k^4}\,\delta\tilde{N}_{\gamma\delta}(\mathbf{k})\ \mathrm{d}^2k + \lim_{S\to\infty}\frac{1}{S}\int\frac{\delta\tilde{N}^*_{\alpha\beta}(\mathbf{k})}{2\pi k^4}$$

$$\times\left\{\int\frac{1}{k'^4}[\delta\hat{\tilde{f}}(\mathbf{k}-\mathbf{k}')A(\mathbf{k},\mathbf{k}') - \delta\hat{\tilde{g}}(\mathbf{k}-\mathbf{k}')B(\mathbf{k},\mathbf{k}')]\,\delta\tilde{N}_{\gamma\delta}(\mathbf{k}')\ \mathrm{d}^2k'\right\}\mathrm{d}^2k + O(\varepsilon_1^4) \tag{68}$$

with

$$\delta\tilde{N}_{\alpha\beta}(\mathbf{k}) = \delta\hat{\tilde{f}}(\mathbf{k})k^2\,\delta_{\alpha\beta} - (\delta\hat{\tilde{f}}(\mathbf{k}) + \delta\hat{\tilde{g}}(\mathbf{k}))k_\alpha k_\beta. \tag{69}$$

It can be shown that this value of the instantaneous compliance equals that obtained by setting $t = 0$ in eqn (62) to within terms of $O(\varepsilon_1^4) + O(\varepsilon\varepsilon_1^3)$.

The long range behavior is obtained by letting the time t tend to infinity in eqn (62). It is readily seen that

$$J_{\alpha\beta\gamma\delta}(\infty) - J_{\alpha\beta\gamma\delta}(0) = \varepsilon\lim_{S\to\infty}\frac{1}{S}\int A_{\alpha\beta\gamma\delta}(\mathbf{k})\ \mathrm{d}^2k + O(\varepsilon\varepsilon_1^3). \tag{70}$$

The slope at the origin $J'_{\alpha\beta\gamma\delta}(0)$ bears some emphasis since knowing $J_{\alpha\beta\gamma\delta}(0), J_{\alpha\beta\gamma\delta}(\infty)$ and $J'_{\alpha\beta\gamma\delta}(0)$ we can approximate the material behavior by a Zener type model. Differentiation of eqn (62) results in

$$J'_{\alpha\beta\gamma\delta}(0) = \varepsilon\lim_{S\to\infty}\frac{1}{S}\int k^2 A_{\alpha\beta\gamma\delta}(\mathbf{k})\ \mathrm{d}^2k + O(\varepsilon\varepsilon_1^3). \tag{71}$$

3.4. *Evaluation of the dissipation*

From definition (64) it follows that $A_{\alpha\beta\gamma\delta} = O(\varepsilon_1^2)$. Hence, the difference between the instantaneous and the long-range compliances is usually very small. This can be verified directly from eqn (70), which yields

$$J_{\alpha\beta\gamma\delta}(\infty) - J_{\alpha\beta\gamma\delta}(0) = O(\varepsilon\varepsilon_1^2). \tag{72}$$

For example, if the fluctuations in the material properties are such that $\varepsilon_1 = 0.1$ and the coupling parameter is $\varepsilon = 0.01$ then

$$J_{\alpha\beta\gamma\delta}(\infty) - J_{\alpha\beta\gamma\delta}(0) = O(10^{-4})$$

i.e. the creep effect is very small. Nevertheless, this small effect may become significant in some cases of interest such as vibrating systems. This point is addressed in Appendix B.

3.5. *Particular case of circularly symmetric spatial correlations*

We denote by

$$\begin{aligned} P_{ff}(\mathbf{k}) &= \delta\hat{f}^*(\mathbf{k})\,\delta\hat{f}(\mathbf{k}) \\ P_{fg}(\mathbf{k}) &= \delta\hat{f}^*(\mathbf{k})\,\delta\hat{g}(\mathbf{k}) \end{aligned} \tag{73}$$

the cross power spectra of f and g and use similar notation for the power spectra of other functions. The power spectra are the Fourier transforms of the spatial correlation functions. Then from eqns (70) and (64) we have

$$\begin{aligned} J_{\alpha\beta\gamma\delta}(\infty)-J_{\alpha\beta\gamma\delta}(0) = \varepsilon \lim_{S\to\infty}\frac{1}{S}\int [&\delta_{\alpha\beta}\delta_{\gamma\delta}P_{cc}-2\delta_{\alpha\beta}\delta_{\gamma\delta}P_{ch}+(2\delta_{\alpha\beta}\delta_{\gamma\delta}-\delta_{\alpha\beta}\xi_\gamma\xi_\beta \\ &-\delta_{\gamma\delta}\xi_\alpha\xi_\beta)P_{cf}+(-\delta_{\alpha\beta}\xi_\gamma\xi_\beta-\delta_{\gamma\delta}\xi_\alpha\xi_\beta)P_{cg}+\delta_{\alpha\beta}\delta_{\gamma\delta}P_{hh} \\ &+(-2\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\beta}\xi_\gamma\xi_\delta+\delta_{\gamma\delta}\xi_\alpha\xi_\beta)P_{hf}+(\delta_{\alpha\beta}\xi_\gamma\xi_\delta+\delta_{\gamma\delta}\xi_\alpha\xi_\beta)P_{hg} \\ &+(\delta_{\alpha\beta}\delta_{\gamma\delta}-\delta_{\alpha\beta}\xi_\gamma\xi_\beta-\delta_{\gamma\delta}\xi_\alpha\xi_\beta+\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta)P_{ff} \\ &+(-\delta_{\alpha\beta}\xi_\gamma\xi_\delta-\delta_{\gamma\delta}\xi_\alpha\xi_\beta+2\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta)P_{fg}+\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta P_{gg}]\,\mathrm{d}^2k. \end{aligned} \tag{74}$$

If spatial correlations are circularly symmetric, the power spectra depend solely on the modulus k of $\mathbf{k}$ and not on the polar angle $\theta = \mathrm{acos}\, k_1/k$. Since $\xi_1 = \cos\theta$, $\xi_2 = \sin\theta$, eqn (74) involves expressions of the form

$$\int a(\theta)P_{ff}(k)\,\mathrm{d}^2k$$

where a is a function of θ, and P_{ff} is a function of the modulus k. Under these conditions it readily follows that:

$$\begin{aligned} \lim_{S\to\infty}\frac{1}{S}\int a(\theta)P_{ff}(k)\,\mathrm{d}^2k &= \left(\int_0^{2\pi} a(\theta)\,\mathrm{d}\theta\right)\lim_{S\to\infty}\frac{1}{S}\int_{k=0}^{\infty} kP_{ff}(k)\,\mathrm{d}k \\ &= \frac{1}{2\pi}\left(\int_0^{2\pi} a(\theta)\,\mathrm{d}\theta\right)\langle\delta f\,\delta f\rangle. \end{aligned} \tag{75}$$

Use will be made of similar results involving P_{fg}, P_{hc}, ..., and of the following identities:

$$\begin{aligned} \int_0^{2\pi}\xi_\alpha\xi_\beta\,\mathrm{d}\theta &= \pi\delta_{\alpha\beta} \\ \int_0^{2\pi}\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta\,\mathrm{d}\theta &= \frac{\pi}{4}(\delta_{\alpha\gamma}\delta_{\beta\gamma}+\delta_{\alpha\beta}\delta_{\beta\gamma}+\delta_{\alpha\beta}\delta_{\gamma\delta}). \end{aligned} \tag{76}$$

From the assumption of circular symmetry, the overall response of the material must be isotropic. Therefore, the macroscopic behavior can be described in terms of two scalar creep functions such that

$$J_{\alpha\beta\gamma\delta} = J_1\delta_{\alpha\beta}\delta_{\gamma\delta}+J_2(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}).$$

Making use of eqns (75) and (76), eqn (74) reduces to

$$\begin{aligned} J_{\alpha\beta\gamma\delta}(\infty)-J_{\alpha\beta\gamma\delta}(0) &= (J_1(\infty)-J_1(0))\delta_{\alpha\beta}\delta_{\gamma\delta}+(J_2(\infty)-J_2(0))\,(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}) \\ &= \varepsilon E(\delta c,\delta h,\delta f,\delta g) \end{aligned} \tag{77}$$

where

$$\begin{aligned} J_1(\infty)-J_1(0) = \varepsilon\{&\langle\delta c\delta c\rangle-2\langle\delta c\delta h\rangle+\langle\delta c\delta f\rangle-\langle\delta c\delta g\rangle+\langle\delta h\delta h\rangle-\langle\delta h\delta f\rangle \\ &+\langle\delta h\delta g\rangle+\tfrac{1}{8}\langle\delta f\,\delta f\rangle-\tfrac{3}{4}\langle\delta f\,\delta g\rangle+\tfrac{1}{8}\langle\delta g\delta g\rangle\} \\ J_2(\infty)-J_2(0) = \varepsilon\{&\tfrac{1}{8}\langle\delta f\,\delta f\rangle+\tfrac{1}{4}\langle\delta f\,\delta g\rangle+\tfrac{1}{8}\langle\delta g\delta g\rangle\}. \end{aligned}$$

As can be seen, to within first-order terms in ε, the long-range behavior is determined by the cross correlations of the fluctuations in the properties of the material.

The initial slope $J'_{\alpha\beta\gamma\delta}(0)$ of the creep function $J_{\alpha\beta\gamma\delta}$ provides a first approximation to the global viscoelastic behavior. This slope is given by eqn (71) which involves terms of the form

$$I = \lim_{S\to\infty} \frac{1}{S} \int k^2 a(\theta)\, \delta\hat{f}^*(\mathbf{k})\, \delta\hat{f}(\mathbf{k})\ \mathrm{d}^2 k. \tag{78}$$

But using the fact that $(\partial\delta f/\partial x_j)(\mathbf{k}) = \mathrm{i}k_j\,\delta f(\mathbf{k})$, eqn (78) can be written as

$$\begin{aligned} I &= \lim_{S\to\infty} \frac{1}{S} \int a(\theta)\, \delta\hat{f}^*_{,j}(\mathbf{k})\, \delta\hat{f}_{,j}(\mathbf{k})\ \mathrm{d}^2 k \\ &= \lim_{S\to\infty} \frac{1}{S} \int a(\theta) P_{f_{,j}f_{,j}}(\mathbf{k})\ \mathrm{d}^2 k. \end{aligned} \tag{79}$$

If we further assume that the spatial correlation functions of the gradients $\delta f_{,j}$, $\delta h_{,j}\ldots$ are circularly symmetric, and proceeding as in the derivation of eqn (75) we obtain

$$\lim_{S\to\infty} \frac{1}{S} \int a(\theta) P_{f_{,j}f_{,j}}(k)\ \mathrm{d}^2 k = \frac{1}{2\pi}\left(\int_0^{2\pi} a(\theta)\,\mathrm{d}\theta\right)\langle \delta f_{,j}\,\delta f_{,j}\rangle. \tag{80}$$

It is interesting to note that this expression involves the correlations of the fluctuation gradients. Finally we have

$$J'_{\alpha\beta\gamma\delta}(0) = -\varepsilon \sum_{j=1}^{2} E(\delta c_{,j}, \delta h_{,j}, \delta f_{,j}, \delta g_{,j})$$

where the function E (., ., ., .) is defined in eqn (77)

Similarly, it is readily shown that the nth outer derivative of $J_{\alpha\beta\gamma\delta}$ involves the nth order partial derivatives of the fluctuations δc, δh, δf, δg.

4. CONCLUSIONS

A proof has been given of the fact that the effective behavior of a linear thermoelastic solid with heterogeneous microstructure subjected to prescribed stresses on remote boundaries corresponds to that of a linear viscoelastic material. By performing a perturbation expansion with respect to two suitably chosen small parameters, first-order approximations to the creep functions of a two-dimensional solid with random microstructure are provided in closed form. These results incorporate the statistical information afforded by the two-point correlation functions of the microscopic fluctuations of material properties.

The relaxation time for thermoelastic damping is typically rather small, of the order of the square of the characteristic length of the microheterogeneities. However, dissipation due to thermoelastic damping can be substantial in dynamic processes whose characteristic duration is of the order of the thermal relaxation time. In particular, thermoelastic effects can be expected to significantly damp the high frequency contents of elastic waves.

Acknowledgements—The authors gratefully acknowledge support from the Brown University NSF Materials Research Laboratory.

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APPENDIX A

Some basic properties of the Fourier transform are recorded next. The Fourier transform of a function $f(\mathbf{x})$ of the spacial coordinates $\mathbf{x} \equiv (x_1, x_2, x_3)$ is defined as

$$\hat{f}(\mathbf{k}) = \left(\frac{1}{2\pi}\right)^{3/2} \int f(\mathbf{x})\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, d^3x. \tag{A1}$$

For instance, if $f(\mathbf{x}) = c = \text{const.}$ one has

$$\hat{f}(\mathbf{k}) = (2\pi)^{3/2} c\delta(\mathbf{k}) \tag{A2}$$

where $\delta(\mathbf{k})$ signifies the Dirac delta distribution at the origin.

The following standard results are used throughout the paper:

$$\begin{aligned} \langle fg \rangle &= \lim_{V\to\infty} \frac{1}{V} \int \hat{f}_V^*(\mathbf{k}) \hat{g}_V(\mathbf{k})\, d^3k \\ \widehat{fg}(\mathbf{k}) &= \left(\frac{1}{2\pi}\right)^{3/2} \int \hat{f}(\mathbf{k}-\mathbf{k}') \hat{g}(\mathbf{k}')\, d^3k'. \end{aligned} \tag{A3}$$

In the first identity f_V and g_V signify the restrictions of f and g to a closed compact subset V of R^3 containing the origin. In terms of the limiting process indicated in eqn (A3a), f_V and g_V are to be regarded as members of two nets of functions obtained by letting the domain V cover the whole space. For example V can be taken to be the ball of radius R centered at the origin and the limit in eqn (A3) reduces to letting $R \to \infty$. It is implicitly assumed that due to statistical disorder the result is independent of the choice of origin. For simplicity of notation, the subindex V in the integrand of eqns (A3) is dropped throughout the paper.

APPENDIX B

Dissipation in a vibrating process

For simplicity, here we consider a one-dimensional solid obeying a creep law of standard Zener type

$$J(t) = J(\infty) - (J(\infty) - J(0))\, e^{-\lambda t}. \tag{B1}$$

The uniaxial stress σ is related to the strain ε by

$$\sigma(t) = \varepsilon * dJ(t) \tag{B2}$$

If a cyclic stress history

$$\sigma = \sigma_0 \cos \omega t \tag{B3}$$

is prescribed, the energy dissipated per cycle is given by

$$\begin{aligned} w &= \pi\sigma_0^2 J'(0) \frac{\omega}{\lambda^2 + \omega^2} \\ &= \pi\sigma_0^2 (J(\infty) - J(0)) \frac{\omega\lambda}{\omega^2 + \lambda^2}. \end{aligned} \tag{B4}$$

To a good approximation, if $J(\infty) - J(0)$ is small, the maximum stored elastic energy is given by

$$E = \tfrac{1}{2}J(0)\sigma_0^2. \tag{B5}$$

The frequency dependent ratio

$$\frac{w}{E}(\omega) = 2\pi\frac{J(\infty)-J(0)}{J(0)}\,\frac{\omega\lambda}{\omega^2+\lambda^2} \tag{B6}$$

represents the part of the elastic energy E which is dissipated into heat per cycle. This ratio attains its maximum value when $\omega = \lambda$

$$\frac{w}{E}(\lambda) = \pi\frac{J(\infty)-J(0)}{J(0)}. \tag{B7}$$

The fraction D/E of elastic energy converted into heat per unit time results from dividing W/E by the period $2\pi/\omega$

$$\frac{D}{E}(\omega) = \frac{J(\infty)-J(0)}{J(0)}\,\frac{\omega^2\lambda}{\omega^2+\lambda^2}. \tag{B8}$$

As $\omega \to \infty$ this quantity tends to an asymptotic limit

$$\frac{D}{E}(\infty) = \lambda\frac{J(\infty)-J(0)}{J(0)} = \frac{J'(0)}{J(0)}. \tag{B9}$$

For $\omega = \lambda$ we have

$$\frac{D}{E}(\lambda) = \frac{1}{2}\,\frac{J'(0)}{J(0)}. \tag{B10}$$

It was noted in Section 3 that $J'(0)$ is inversely proportional to the square of some characteristic length L of the heterogeneities. Consequently, for small L the fraction D/E may become significant, even if $J(\infty)-J(0)$ is small.